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Sequences in a weighted graph and characterization of partial trees

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Abstract— In a weighted graph, the arcs are mainly classified into α, β and δ. In this article, some sequences in weighted graphs are introduced. These concepts are based on the above classification. Characterizations of partial trees and some necessary conditions are obtained. It is shown that β sequence of a partial tree is a zero sequence.

AMS subject classification: 03E72, 03E75, 05C22, 05C38.

Index Terms— α- sequence, β- sequence, strong sequence, partial trees

1 INTRODUCTION

GRAPH theory has now become a major branch of mathematics and it is generally regarded as a branch of combinatorics. Graph is a widely used tool for solving a combinatorial problem in differene areas such as geometry, algebra, number theory, topology, optimization and computer science. Most important thing which is to be noted is that, any problem which can be solved by any graph technique can only be modeled as a weighted graph problem. Distance and central concepts play an important role in applications related with graphs and weighted graphs. Several authors including Bondy and Fan [2, 3, 4], Broersma, Zhang and Li [12], Sunil Mathew and M S Sunitha [7, 8, 9, 10, 11, 12] introduced many connectivity concepts in weighted graphs following the works of Dirac [5] and Grotschel [6].

In this article, we introduce three new sequences in weighted graphs. These concepts are derived by using the notion of connectivity in weighted graphs. In a weighted graph model, for example, in an information net work or in an electric circuit, the reduction of flow between the pairs of nodes is more relevant and may frequently occur than total disruption of the entire net work [9, 10, 14]. Finding the cenre of a weighted graph is useful in facility location problems where the goal is to minimize the distance to the facility. For example, placing a hospital at a central point reduces the longest distance the ambulance has to travel. This concept is our motivation. As weighted graphs are generalized strctures of graphs, the concepts introduced in this article also generalize the classic ideas in graph theory.

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2 PRELIMINARIES

A weighted graph G: (V, E, W) is a graph in which every arc e is assigned a nonnegative real number w(e) called the weight of e [1]. In a weighted graph G: (V, E, W) the strength of a path P = v_0 e_1v_1 e_2 v_2 …e_n v_n is defined and denoted by S(P) = min{w(e_1), w(e_2), w(e_3), …, w(e_n)} [10]. The strength of connectedness of a pair of nodes u and v in G is defined and denoted by $CONN_G$(u, v) = max{S(P)/ P is a u- v path in G} [9]. A u – v path P is called a strongest u – v path if S(P) = $CONN_G$(u, v) [9]. A node w is called a partial cut node (p cut node) of G if there exists a pair of nodes u, v in G such that u ≠ v ≠ w and $CONN_{G-w}$(u, v) < $CONN_G$(u, v) [9]. A graph without p- cut nodes is called a partial block (p- block) [9]. It is also proved in [9] that a node w in a weighted graph G is a p-cut node of G if and only if w is an internal node of every maximum spanning tree of G. A connected weighted graph G: (V, E, W) is called a partial tree (p- tree) if G has a spanning subgraph F: (V, E′, W′) which is a tree, where for all arcs e = (u, v) of G which are not in F, $CONN_G$ (u, v) > w(e) [9]. An arc e = (u, v) is called α- strong if $CONN_{G-e}$(u, v) < w(e) and β- strong if $CONN_{G-e}$(u, v) = w(e) and a δ- arc if $CONN_{G-e}$(u, v) > w(e). An arc is called strong if it is either α- strong or β- strong [9].

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3 SEQUENCES IN A WEIGHTED GRAPH

In this section, we define three types of sequences. These are based on α and β arcs in a weighted graph.

3.1 Definition

Let G (V, E, W) be a connected weighted graph with |V| = p. Then a finite sequence α_s (G) = (n_1, n_2, n_3, ..., n_p) ϵ Z_0^{+p} is called theα- sequence of G if n_i = number of α- strong arcs incident on vertex v_i and = 0 if no α- strong arc is incident on v_i.

If there is no confusion regarding G, we use the notation α_s instead of α_s (G).

3.2 Definition

Let G: (V, E, W) be a connected weighted graph with |V| = p.

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Then a finite sequence β_s (G) = (n_1, n_2, n_3, ..., n_p) ϵ Z_0^{+p} is called the β- sequence of G if n_i = number of β- strong arcs incident on vertex v_i and = 0 if no β- strong arc is incident on v_i.

If there is no confusion regarding G, we use the notation β_s instead of β_s (G).

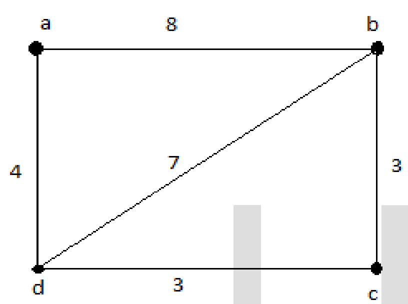
3.3 Definition

Let G: (V, E, W) be a connected weighted graph with |V| = p. Then a finite sequence S_s (G) = (n_1, n_2, n_3, ..., n_p) ϵ Z_0^{+p} is called the strong sequence of G if n_i = number of strong arcs incident on vertex v_i and = 0 if no strong arc is incident on v_i.

If there is no confusion regarding G, we use the notation S_s instead of S_s (G).

3.4 Example

In the following figure, all these sequences are illustrated.



α_s = (1, 2, 0, 1)
β_s = (0, 1, 2, 1)
S_s = (1, 3, 2, 2)

4 Some Necessary Conditions

In this section, we present some necessary conditions which must be satisfied by a partial tree.

4.1 Theorem

If G: (V, E, W) is a partial tree and |V| = p, then α_s(G) ϵ Z^{+p}

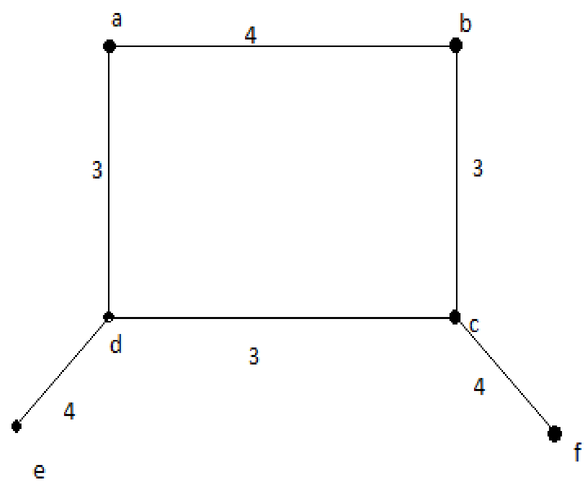
That means all the entries in α_s(G) is at least unity.

Proof:

By definition of α_s(G), it is clear that all of it's elements are greater than or equal to zero. We want to prove that all the elements in α_s(G) are at least unity. Suppose the contrary. Let the i^{th} element in α_s(G) ,say, n_i be zero. Since n_i = 0, the corresponding node v_i will not be incident with any α- strong arc. This will result in the disconnection of the maximum spanning tree F of G, which is a contradiction to the definition of F. So our assumption is wrong and hence all the elements in α_s(G) are at least unity. This completes the proof of the theorem.

The condition in the above theorem is not sufficient as seen from the following example

4.2 Example



In this graph, α_s (G) = (1, 1, 1, 1, 1, 1). But the graph is not a partial tree.

4.3 Theorem

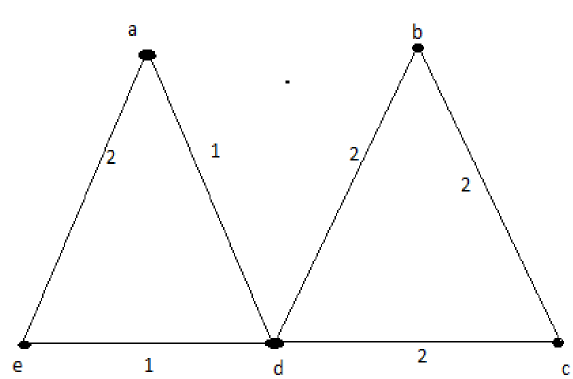
Let G: (V, E, W) be a connected weighted graph such that |V| = p. Let t be a positive integer such that t ≤ p. If α_s (G) contains t elements which are greater than or equal to 2, then G has exactly t partial cut nodes.

Proof:

Let G: (V, E, W) be a partial tree. Let F be the spannoing tree of G with the property given in the definition of partial trees. Then the internal nodes of F are the partial cut nodes of G [12]. Also we know that, if a node is common to more than one α-strong arc, then it is a partial cut node [12]. So the node of G which corresponds to an entry in α_s (G) which is greater than or equal to 2 must be a partial cut node. This completes the proof of the theorem.

If the condition in the above theorem was sufficient, we will be able to identify the partial cutnodes of G with the information about the α- sequence of G. But due to non sufficiency, we can get the number of partial cut nodes of G only. This fact is illustrated in the following example.

4.3 Example



In this graph, node d is a cut node and hence a a partial cut node, but entry corresponding to the α- eequence is 0.

5 Characterization of Partial Trees

In this section, we characterize partial trees using β- sequence of G. By (0), we mean the zero sequence which contains only zeros.

5.1 Theorem

A connected weighted graph G: (V, E, W) is a partial tree if and only if β_s (G) = (0).

Proof:

Let G: (V, E, W) be a connected weighted graph. Suppose that G is a partial tree. If G is a weighted tree, then all the arcs of G are bridges and hence partial bridges. Now an arc e = (u, v) in G is a partial bridge if and only if it is α- strong [12]. Thus all the arcs in G are α- strong. So G has no β- strong arcs and hence β_s (G) = (0).

If G is not a weighted tree, then G has a weighted cycle, say, C. Since G is a partial tree, there exists an arc e = (u, v) such that $CONN_{G\text{-}e}$ (u, v) > w (e), where G-e is the subgraph of G obtained by deleting the arc e from G [12]. That means e is a δ arc. If G-e is a weighted spanning tree of G, all the arcs in G-e are α. Hence G has no β- strong arcs. So β_s (G) = (0). If G-e is not a weighted spanning tree of G, then continue the above procedure of deleting δ arcs from G-e until we get a weighted spanning tree.

Conversely suppose that β_s (G) = (0). We have to prove that G is a partial tree. If G has no cycles, then G is a weighted tree and hence a partial tree. Suppose that G has a cycle, sy, C. Then C wil contain only α- strong and δ arcs. Also note that all arcs of C cannot be α- strong, since otherwise it will contradict the definition of α- strong arcs. Thus there exists atleast one δ arc in C. If we delete e from C, we get a maximum spanning tree of G. If not remove one δ arc from existing weighted cycles from G. Continue this procedure until we get a maximum spanning trer of G. Hence G is a partial tree.

6 Conclusion

In this article, three types of sequences in weighted graphs are introduced. As reduction in strength between two nodes is more important and useful in practical applications than total disconnection of the entire graph, the authors made use of connectivity concepts in defining the sequences. A special interest on characterizing partal tree structure can be seen as they are applied widely. Eventhough this structure has got many characterizations; here we did it in a simpler way using β- sequences.

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